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Algorithms for special integrals of ordinary differential equations

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Abstract. We give new, conceptually simple procedures for calculating special integrals of polynomial type (also known as Darboux polynomials, algebraic invariant curves, or eigenpolynomials), for ordinary differential equations. In principle, the method requires only that the given ordinary differential equation be itself of polynomial type of degree one and any order. The method is algorithmic, is suited to the use of computer algebra, and does not involve solving large nonlinear algebraic systems. To illustrate the method, special integrals of the second, fourth and sixth Painlevé equations, and a third-order ordinary differential equation of Painlevé type are investigated. We prove that for the second Painlevé equation, the known special integrals are the only ones possible.

1. Introduction

The motivation for this article was to find, algorithmically, general integrals of ordinary differential equations such as

$$x^2FF''' = (x^2F' - xF)F'' - F^3F' - 2\mu_1(xF - x^2F') + \mu_2 xF^2 \qquad (1.1)$$

where $' \equiv \mathrm{d}/\mathrm{d}x$. Equation (1.1) arises as a reduction of an integrable system, in this case a $(2+1)$-dimensional sine-Gordon system [1]. Despite not possessing any symmetries to aid in its integration, it possesses a general integral which is a rational expression in F'', F', F and x.

The seeking of such integrals has a long history. Darboux [2] sought rational solutions to first-order, first-degree equations, Δ, and noted that the factors, Q, of the numerator and denominator of the solution satisfy an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}Q - bQ \equiv 0 \mod \Delta. \qquad (1.2)$$

This insight generalizes to higher-order equations, for which in (1.2) one seeks *differential* polynomials, that is, expressions which are polynomial in the variables and the derivative terms. The expression Q is known variously as a Darboux polynomial, algebraic invariant curve, special integral, or eigenpolynomial, amongst other terms [3]. Recent articles have advocated the calculation of Q by substituting arbitrary differential polynomials in (1.2)

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and solving the resulting nonlinear algebraic system for the numerical coefficients, using the theory of Gröbner bases [4, 5]. For an equation such as (1.1), this method requires considerable computational effort. Even if one does know the degrees of F'', F', F and x in Q, namely $2, 3, 4$ and 2, which *a priori* one does not, one would still have hundreds of nonlinear equations for the 180 coefficients to be determined. In the special case of $\mu_2 = 0$, (1.1) was integrated in [6] by assuming an ansatz of the form

$$Q = A(x, F, F')(F'')^2 + B(x, F, F')F'' + C(x, F, F')$$

and solving the resulting overdetermined nonlinear system of partial differential equations for A, B and C, by means of various *ad hoc* simplifying assumptions, such as restricting the dependence of the functions A, B and C. However, an *algorithmic*, computationally feasible solution was sought.

The method developed in this paper reduces the calculation of Q and b for (1.1) to a succession of linear ordinary differential equations of Euler type. This is achieved by first solving for the dependence of A, B and C in (1.1) on F', iterating to obtain their dependence on F, and finally solving an overdetermined system of consistency conditions for their dependence on x. For the examples in this paper at least, these overdetermined systems can be solved straightforwardly. The fact they are overdetermined means integrability conditions and the like can be calculated to simplify them, or in other examples, lead to conditions on the parameters in the original equation for a solution to exist.

We illustrate the ideas involved by calculating special integrals of polynomial type of the second, fourth and sixth Painlevé equations. The abundance of such special integrals, or one-parameter family conditions, for these equations means they are a good test of any method designed to find them. For example, in 1910, Gambier [8] discovered that for $\alpha = \frac{1}{2}$, the second Painlevé equation

$$\Phi_\alpha \equiv y'' - 2y^3 - xy - \alpha = 0 \tag{1.3}$$

has the special integral

$$Q = y' - y^2 - \tfrac{1}{2}x \tag{1.4}$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}Q + 2yQ = y'' - 2y^3 - xy - \tfrac{1}{2}. \tag{1.5}$$

Note that Q is not a general first integral in the sense that any solution of $Q = \kappa$ where κ is an arbitrary constant will satisfy $\Phi_{1/2}$, only a solution of $Q = 0$ will satisfy $\Phi_{1/2}$. We use our method to prove that the second Painlevé equation has special integrals of polynomial type if and only if

$$\alpha = \tfrac{1}{2}(2n + 1) \qquad n \in \mathbb{Z}.$$

The existence of these special integrals leads to solutions of the second Painlevé equation in terms of the Airy function for these parameter values.

Painlevé, and subsequently Fuchs and others around the turn of the century, sought to classify those ordinary differential equations of the form,

$$y'' = F(y', y, x)$$

where F is rational in y' and y and analytic in x, whose solutions have no movable singularities other than poles. The Painlevé equations are six nonlinear ordinary differential equations having these properties whose general solutions cannot be expressed in terms of the known elementary or transcendental functions. The Painlevé transcendents are regarded as nonlinear special functions as they possess several properties analogous to those of the classical special functions. For example, for certain parameter values, the classical transcendental equations have polynomial solutions and solutions expressible in terms of elementary functions, while the Painlevé equations have particular solutions which are rational or expressible in terms of classical transcendental functions. In addition, the classical transcendental functions satisfy recurrence relations while Bäcklund transformations relate solutions of the Painlevé equations. Finally, the Laplace transformation can be used to solve the classical transcendental equations, while the Painlevé equations can be solved by the isomonodromy deformation method (cf [7, 9, 10]).

Special integrals of polynomial type for the Painlevé equations can often be obtained from the isomonodromy deformation method. The idea behind the method is to study the Painlevé equation by expressing it as an integrability condition of a linear system, or Lax pair, which possesses both regular and irregular singular points. It is often the case that for certain parameter values the formal series solution, or Frobenius expansion, about a certain regular singular point breaks down. (Within the terminology of Frobenius this occurs when the roots of the indicial equation differ by an integer). These parameter values are the same as those characterizing particular integrals of the associated Painlevé equation, and indeed the nature of the formal solution of the linear system restricts the variables to be solutions of the particular integrals of the associated Painlevé equation. In many cases the full set of known particular integrals arises in this way, however this is not always true [10]. Examples exist where only part of the set is characterized in this way. In these cases there are definite indications that the 'missing' integrals are characterized in some other part of the linear system, but such calculations require experience with the isomonodromy deformation method, and results may not be as easy to obtain as the method described here. Further, it is a non-trivial problem to obtain the Lax pair for an ordinary differential equation of Painlevé type in general.

Equations of Painlevé type have become important in recent years because they arise as reductions of systems solvable by inverse scattering, and their exact analytic solutions are important as they can be used to generate exact solutions of nonlinear integrable systems [7, 11]. Painlevé equations have many physical applications, including asymptotics of nonlinear evolution equations, correlation functions in the XY model, the two-dimensional Ising model, statistical mechanics, plasma physics, resonant oscillations in shallow water, convective flows with viscous dissipation, Görtler vortices in boundary layers, nonlinear waves, polyelectrolytes, general relativity, nonlinear optics and fibre optics, quantum gravity and quantum field theory. References for these applications can be found in [7, section 7.1.6] and [12, p 119]. Since many of the known solutions of the Painlevé equations arise as solutions of special integrals of polynomial type occurring for certain values of the parameters, a systematic procedure that can generate all such integrals is important.

Kolchin and his group knew that the first Painlevé equation has no special integrals of polynomial type [13], the proof being recorded in a handwritten manuscript† (see also [14]).

In seeking integrals more general than those of rational type, historically 'elementary' and 'Liouvillian extensions' have been sought, that is, expressions that also involve radical, exponential and logarithmic terms, and their integrals with respect to the independent variable. Prelle and Singer proved that to find elementary first integrals for general algebraic differential equations, it was sufficient to seek them in a certain form, and they demonstrate the central role the special integrals or Darboux polynomials have in the calculation of elementary first integrals in their examples [15]. More recently, there has been interest in

† ELM is indebted to Michael Singer for a copy of this manuscript.

the relationship between the existence of special integrals and the differential Galois group for linear differential equations [3]. Theorems proving bounds on the degree (with respect to y') of polynomial type integrals for certain sorts of differential systems are known (cf [15] and references therein, and [16]). The examples given in the present paper show that in general any such bound depends necessarily on the numerical coefficients and parameters appearing in the equation.

The method for the calculation of special integrals demonstrated here is algorithmic, simple in concept, and applicable (in principle) to any differential equation of polynomial type in which the highest-order derivative term occurs to highest power one. The method is simple enough to be able to be generalized to function spaces other than polynomial, but the function space used would depend on the equation to be integrated.

2. Integrals of polynomial type

Here, we briefly review the main facts concerning special integrals of polynomial type, and fix our notation. For simplicity, we assume an ordinary differential equation of order two and degree one, and thus an integral of order one. However, all the results in this section generalize to equations of arbitrary order and degree one. (Recall that the order of a differential equation is the highest number of times a dependent variable has been differentiated in the equation, while the degree of the equation is the highest power of the highest derivative term occurring in the equation).

We assume the integral Q to be of the form

$$Q = \sum_{k=0}^{N} q_k(x, y)(y')^k \tag{2.1}$$

where $N > 0$, $q_N \neq 0$ and where the q_k are assumed to be polynomial with respect to y. For the sake of simplicity, suppose further that Q is irreducible, that is, Q has no more than one factor depending on y', and let the equation studied be of the form

$$\Phi \equiv y'' - W(x, y, y') = 0$$

where W is a polynomial with respect to y and y'. By definition, the special integral Q satisfies a relation

$$D(x, y, y', y'')\frac{\mathrm{d}Q}{\mathrm{d}x} - B(x, y, y', y'')Q = H(x, y, y', y'')(y'' - W(x, y, y')) \tag{2.2}$$

where D, B and H are polynomial with respect to all variables except perhaps x, and where $Q = 0$ and $H = 0$ have only trivial solutions in common. This implies in particular that Q does not divide H. The reason for this assumption is that we want any solution of $Q = 0$ to be a solution of Φ.

Let Q_x denote the partial derivative of Q with respect to x, that is, the derivative of Q with respect to its explicit dependence on x, regarding x, y and y' as separate independent variables, and similarly Q_y and $Q_{y'}$. We then have the identity

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = Q_x + y'Q_y + y''Q_{y'} \tag{2.3}$$

showing that $\mathrm{d}Q/\mathrm{d}x$ is linear in y''. We can now considerably simplify the problem (2.2) to be solved. All occurrences of y'' in D and B can be absorbed into H by subtracting suitable multiples of Φ. Since Q does not depend on y'', we have by comparing powers of

y'' on both sides of the equation (2.2) that H does not depend on y'', so that we need only solve

$$D(x, y, y')\frac{\mathrm{d}Q}{\mathrm{d}x} - B(x, y, y')Q = H(x, y, y')(y'' - W(x, y, y')). \tag{2.4}$$

Further, considering coefficients of y'' on both sides of the equation (2.4) leads to the identity, $DQ_{y'} = H$, hence it sufficient to solve

$$D(x, y, y')\left(Q_x + y'Q_y + W(x, y, y')Q_{y'}\right) = B(x, y, y')Q. \tag{2.5}$$

Now, since Q does not divide H, it does not divide D, so it must be true that Q divides $Q_x + y'Q_y + W(x, y, y')Q_{y'}$, using the assumption that Q is irreducible (it is a well known fact from algebra that if f, g and h are polynomials with f irreducible, such that f divides gh, then f must divide one of g, h). Thus, it is sufficient to solve the problem

$$Q_x + y'Q_y + W(x, y, y')Q_{y'} = b(x, y, y')Q. \tag{2.6}$$

For the more general $\Phi \equiv A(x, y, y')y'' - W(x, y, y')$, the same arguments yield that it is sufficient to solve

$$AQ_x + Ay'Q_y + WQ_{y'} = b(x, y, y')Q. \tag{2.7}$$

In the next lemma, we note that our search for polynomial integrals will yield every solution infinitely many times!

Lemma 2.1. If Q is an integral of polynomial type, then so is Q^p for any $p \in \mathbb{N}$, $p \neq 0$.

Proof. Suppose Q satisfies (2.7). Inserting Q^p in the left-hand side of (2.7) we obtain that Q^p satisfies

$$A(Q^p)_x + Ay'(Q^p)_y + W(x, y, y')(Q^p)_{y'} = p \cdot b(x, y, y')Q^p.$$

Thus, any solution Q of (2.7) will imply infinitely many 'redundant solutions', Q^p. Indeed, a function of an integral will be an integral, although not necessarily of polynomial type. □

Irreducible special integrals of polynomial type are not unique. Given several such integrals, it is sometimes possible to combine them to obtain a general integral. A summary of known results appears in [5].

Although we assumed Q to be irreducible, this is not actually a restriction. It is simple to show that if Q_1 and Q_2 are integrals, then so is Q_1Q_2. The converse is also true; if Q_1Q_2 is an integral, then so are Q_1 and Q_2 (provided both factors contain the highest derivative term); cf [3]. Thus it suffices to find the irreducible integrals.

Before turning to the mechanics of finding special integrals of polynomial type of particular equations, we record in the following table the so-called leading-order constraints, in the case $A_{y'} = 0$. We use (2.1) and (2.7) and set

$$b(x, y, y') = \sum_{k=0}^{M} b_k(x, y)(y')^k \qquad W = \sum_{k=0}^{R} w_k(x, y)(y')^k.$$

Comparing leading powers of y' on both sides of (2.7) yields constraints on M, q_N, and b_M, which are given in table 1.

Table 1. Leading-order constraints for integrals of $A(x, y)y'' - W(x, y, y') = 0$. $N = \deg(Q, y')$, $M = \deg(b, y')$, $R = \deg(W, y')$.

R	Constraints	Possibilities for M
0, 1	(i) $q_{N,y} = 0$	$M = 0$
	(ii) $Aq_{N,y} = q_N b_M$	$M = 1$
2	(i) $Aq_{N,y} + Nw_R q_N = 0$	$M = 0$
	(ii) $Aq_{N,y} + Nw_R q_N = b_M q_N$	$M = 1$
$\geqslant 3$	$Nw_R = b_M$	$M = R - 1$

3. Calculations for the second Painlevé equation

In this section we show how to generate systematically all integrals of polynomial type for the second Painlevé equation,

$$\Phi_\alpha \equiv y'' - 2y^3 - xy - \alpha = 0. \tag{1.3}$$

Considering table 1, for the second Painlevé equation we have $R = 0$, that is, $W_{y'} = 0$, and $A = 1$, so there are two cases to consider, (i) $M = 0$ and $q_{N,y} = 0$, and (ii) $M = 1$ and $q_{N,y} = b_M q_N$. The second case has no solutions for q_N polynomial in y. Before considering case (i) in more detail for the second Painlevé equation, we note the following lemma which we use to simplify the calculation.

Lemma 3.1. If $q_{N,y} = 0$, it is sufficient to consider $q_N \equiv 1$.

Proof. Assume that Q satisfies (2.6). Inserting $g(x)Q$ in the left-hand side of (2.6) we obtain that $g(x)Q$ satisfies

$$(gQ)_x + y'(gQ)_y + W(x, y, y')(gQ)_{y'} = \big(g'(x)/g(x) + b(x, y, y')\big)(gQ)$$

so any dependence of q_N on x can be absorbed into $b(x, y, y')$. □

Thus, to find special integrals of the second Painlevé equation of polynomial type, it is sufficient to solve

$$Q_x + y'Q_y + (2y^3 + xy + \alpha)Q_{y'} = b(x, y)Q \tag{3.1}$$

with Q of the form

$$Q = \sum_{k=0}^{N} q_k(x, y)(y')^k \qquad q_N \equiv 1.$$

Reading off coefficients of powers of y' in (3.1) yields the system

$$q_{N-1,y} = b \tag{3.2$_{N-1}$}$$

$$\vdots$$

$$q_{N-k,y} = bq_{N-k+1} - (N-k+2)(2y^3 + xy + \alpha)q_{N-k+2} - q_{N-k+1,x} \tag{3.2$_{N-k}$}$$

$$\vdots$$

$$q_{0,y} = bq_1 - 2(2y^3 + xy + \alpha)q_2 - q_{1,x} \tag{3.2$_0$}$$

$$0 = bq_0 - (2y^3 + xy + \alpha)q_1 - q_{0,x}. \tag{3.2$_{cc}$}$$

Setting

$$b = \sum_{j=0}^{m} \beta_j(x) y^j$$

we have from (3.2_{N-1}) that

$$q_{N-1} = \sum_{j=0}^{m} \frac{\beta_j(x)}{j+1} y^{j+1} + H_{N-1}(x).$$

This is then inserted into (3.2_{N-2}), which is integrated symbolically to obtain an expression for q_{N-2} in terms of the β_j, $j = 0, \ldots, m$ and H_{N-1}, with $H_{N-2}(x)$ being the function of integration. Continuing in this way, we can express the $q_{N-1}, \ldots, q_0$ in terms of the β_j with $N-1$ functions of integration, the H_k, $k = 0, \ldots, N-1$. Then, inserting the expressions for q_0 and q_1 into (3.2_{cc}), we obtain a system, $\mathcal{C}$, of *consistency conditions* for the β_j and the H_k, by setting the coefficients of the various powers of y to zero.

We now determine the possible values for m, the degree of b with respect to y. If $m \neq 0$, the degree of q_k with respect to y is $(N-k)(m+1)$. Then (3.2_{cc}) implies that in order to balance powers of y, we must have that $m + N(m+1) = 3 + (m+1)(N-1)$ or $m = 1$. If $m = 0$, then N must be an even integer. For both cases, $m = 0$ or 1, we obtain that $\deg(q_k, y) \leqslant 2(N-k)$.

3.1. *The simplest examples*

The case $N = 1$. We show this case explicitly both to show how trivial is the calculation, and how the value of α arises naturally as a consistency condition. Inserting $Q = q_0(x, y) + y'$ in (3.1) and equating coefficients of powers of y' to zero yields

$$q_{0,y} = b \tag{3.3a}$$

$$q_{0,x} + 2y^3 + xy + \alpha = bq_0. \tag{3.3b}$$

We have $b = \beta_0(x) + \beta_1(x)y$, implying $q_0 = \beta_0(x)y + \frac{1}{2}\beta_1(x)y^2 + H_0(x)$. Inserting this in (3.3*b*), we obtain $\mathcal{C}$:

$$2 = \beta_1^2/2 \qquad \beta_0'(x) + x = H_0(x)\beta_1 + \beta_0^2$$

$$\beta_1'(x) = 3\beta_0\beta_1/2 \qquad \beta_0(x)H_0(x) = \alpha + H_0'(x).$$

Thus

$$b(x, y) = \pm 2y \qquad Q = y' \pm y^2 \pm x/2 \qquad \alpha = \mp\tfrac{1}{2}$$

which are the integrals for $\Phi_{\pm 1/2}$ known to Gambier.

The case $N = 2$. Performing the same calculation for $N = 2$ yields only the squares of the two integrals obtained in the $N = 1$ case above.

The case $N \geqslant 3$. The best strategy for solving the consistency conditions $\mathcal{C}$ is to begin with the coefficient of the highest power of y in (3.2_{cc}), since that involves the fewest number of indeterminants, and to work down. We obtain the polynomial integral for $\Phi_{-3/2}$ to be

$$Q_{-3/2} = (y')^3 + (y^2 + \tfrac{1}{2}x)(y')^2 - \left[y^2 + \tfrac{1}{2}x)^2 - 4y\right]y' - (y^2 + \tfrac{1}{2}x)^3 + 4y(y^2 + \tfrac{1}{2}x) - 2.$$

while the integral of $\Phi_{3/2}$ is

$$Q_{3/2} = (y')^3 - (y^2 + \tfrac{1}{2}x)(y')^2 - \left[(y^2 + \tfrac{1}{2}x)^2 + 4y\right]y' + (y^2 + \tfrac{1}{2}x)^3 + 4y(y^2 + \tfrac{1}{2}x) + 2.$$

Similarly, the integral for $\Phi_{-5/2}$ is

$$Q_{-5/2} = (y')^5 + (y^2 + \tfrac{1}{2}x)(y')^4 - 2\left[(y^2 + \tfrac{1}{2}x)^2 - 6y\right](y')^3$$
$$-2(y^2 + \tfrac{1}{2}x)\left[(y^2 + \tfrac{1}{2}x)^2 - 6y\right](y')^2 + \left\{\left[(y^2 + \tfrac{1}{2}x)^2 - 6y\right]^2 + 2x\right\} y'$$
$$+(y^2 + \tfrac{1}{2}x)^5 - 12y(y^2 + \tfrac{1}{2}x)^3 + 4y^4 + 26xy^2 - 32y + \tfrac{5}{2}x^2 .$$

The formulae for the integrals become considerably more complex with increasing N, but the calculation of them is quite straightforward.

Solutions to the $Q = 0$, and hence to the second Painlevé equation, are expressible in terms of Airy functions [7, 8, 17, 18].

3.2. Transformations connecting the integrals

The second Painlevé equation has two well known Bäcklund transformations

$$y(x; -\alpha) = -y(x; \alpha) \tag{3.4a}$$

$$y(x; \alpha + 1) = -y(x; \alpha) - \frac{1 + 2\alpha}{2y(x; \alpha)' + 2y(x; \alpha)^2 + x} \qquad \alpha \neq -\tfrac{1}{2}. \tag{3.4b}$$

A suitable combination of these two Bäcklund transformations yields the 'inverse' transformation

$$y(x; \alpha - 1) = -y(x; \alpha) - \frac{1 - 2\alpha}{2y(x; \alpha)' - 2y(x; \alpha)^2 - x} \qquad \alpha \neq \tfrac{1}{2}. \tag{3.5}$$

Note that the denominators of these transformations are proportional to $Q_{\pm 1/2}$.

Suppose, for example, y satisfies $\Phi_{3/2}$. Using (3.5) one obtains Y which satisfies $\Phi_{1/2}$. Then, inserting the expression for Y into the integral $Q_{1/2}$ yields a rational expression whose numerator is $Q_{3/2}$. Conversely, suppose y satisfies $\Phi_{1/2}$. Substituting (3.4b) in $Q_{3/2} = 0$ with $\alpha = \frac{1}{2}$ yields $Q_{1/2}Q^2_{-1/2} = 0$; since $Q_{-1/2}$ is the denominator of the Bäcklund transformation and is therefore non-zero, we have obtained $Q_{1/2} = 0$.

More generally, it can be seen that Bäcklund transformations 'lift' to the integrals, but operate in the reverse direction; a transformation which sends solutions of Φ_{α_1} to solutions of Φ_{α_2} will map an equation $Q_{\alpha_2} = 0$ to an equation $Q_{\alpha_1} = 0$ (after removal of non-zero factors). Note that since the Bäcklund transformations are of rational type, we obtain mappings between integrals of polynomial type.

It follows that there is an integral of polynomial type for Φ_α for every α of the form, $(2n + 1)/2$, $n \in \mathbb{Z}$. Further, the degree of $Q_{(2n+1)/2}$ with respect to y' is $|2n + 1|$. This example shows that any bound on the degrees of special integrals depends necessarily on the numerical coefficients and parameters appearing in the equation.

3.3. Recursion formulae for the general case

The method of integration outlined above is highly iterative. Indeed, recalling that $\deg(q_k, y) \leqslant 2(N - k)$, and setting

$$q_k(x, y) = P_k(\beta_1)y^{2(N-k)} + S_k(\beta_0, \beta_1)y^{2(N-k)-1} + T_k(\beta_0, \beta_1, x, H_{N-1})y^{2(N-k)-2}$$
$$+F_k(\beta_0, \beta_1, N, H_{N-1}, \alpha)y^{2(N-k)-3} + \mathcal{E}_k \tag{3.6}$$

where $\mathcal{E}_i$ are the lower-order terms with respect to y, the calculation yields the (descending) recursion formulae

$$P_k = \frac{1}{2(N-k)}\left(\beta_1 P_{k+1} - 2(k+2)P_{k+2}\right) \tag{3.7a}$$

$$S_k = \frac{1}{2(N-k)-1}\left(\beta_1 S_{k+1} - 2(k+2)S_{k+2} + \beta_0 P_{k+1} - \frac{\mathrm{d}}{\mathrm{d}x}P_{k+1}\right) \tag{3.7b}$$

$$T_k = \frac{1}{2(N-k)-2}\left(\beta_1 T_{k+1} - 2(k+2)T_{k+2} - \frac{\mathrm{d}}{\mathrm{d}x}S_{k+1} - (k+2)xP_{k+2} + \beta_0 S_{k+1}\right) \tag{3.7c}$$

$$F_k = \frac{1}{2(N-k)-3}\left(\beta_1 F_{k+1} - (k+2)\,(2F_{k+2} + xS_{k+2} + \alpha P_{k+2}) - \frac{\mathrm{d}}{\mathrm{d}x}T_{k+1} + \beta_0 T_{k+1}\right) \tag{3.7d}$$

with initial data

$$P_N = 1 \qquad P_{N-1} = \tfrac{1}{2}\beta_1 \tag{3.8a}$$

$$S_N = 0 \qquad S_{N-1} = \beta_0 \tag{3.8b}$$

$$T_N = 0 \qquad T_{N-1} = H_{N-1}(x) \tag{3.8c}$$

$$F_N = 0 \qquad F_{N-1} = 0. \tag{3.8d}$$

Inserting (3.6) in (3.2_{cc}) and then setting to zero the coefficients of y yields the consistency conditions

$$\beta_1 P_0 - 2P_1 = 0 \tag{3.9a}$$

$$\beta_0 P_0 + \beta_1 S_0 - 2S_1 - P_0' = 0 \tag{3.9b}$$

$$\beta_0 S_0 + \beta_1 T_0 - 2T_1 - xP_1 - S_0' = 0 \tag{3.9c}$$

$$\beta_0 T_0 + \beta_1 F_0 - 2F_1 - xS_1 - \alpha P_1 - T_0' = 0 \tag{3.9d}$$

where $' \equiv \mathrm{d}/\mathrm{d}x$. These relations will be used in the proof of the following theorem.

Theorem 3.2. There exist special integrals of polynomial type for the second Painlevé equation, (1.3), if and only if

$$\alpha = \tfrac{1}{2}(2n+1) \qquad n \in \mathbb{Z}.$$

Proof. We use the recursion relations above to obtain a formula connecting α and N. We then examine the possibilities. We may assume that $N > 3$ since we have already performed the calculation for $N \leqslant 3$. We begin by solving the recursion relation for P_k. Define

$$p(z) = P_N + P_{N-1}z + \cdots + P_0 z^N$$

that is

$$P_j = \frac{1}{2\pi\mathrm{i}}\int_C z^{j-N-1}p(z)\,\mathrm{d}z$$

where C is a suitably small contour encircling the origin in $\mathbb{C}$ once in an anti-clockwise direction. Note that the conditions (3.8*a*) imply that $P_k = 0$ for $k > N$, while condition (3.9*a*) is equivalent to $P_{-1} = 0$, which then implies $P_k = 0$, $k \leqslant -1$. Using $j\int_C z^{j-1} f(z)\,\mathrm{d}z = -\int_C z^j f'(z)\,\mathrm{d}z$, one obtains

$$(k+2)P_{k+2} - \tfrac{1}{2}\beta_1 P_{k+1} + (N-k)P_k$$
$$= \frac{1}{2\pi\mathrm{i}}\int_C z^{k-N}[(1-z^2)p'(z) + (Nz - \tfrac{1}{2}\beta_1)p(z)]\,\mathrm{d}z \qquad k = 0,\ldots,N-2.$$

Now, since p is a polynomial of degree N, $(1-z^2)p'(z) + (Nz - \frac{1}{2}\beta_1)p(z)$ is a polynomial of degree $N+1$, and the above calculation implies the coefficients of z^k, where $k = 1,\ldots,N-1$, are zero. Therefore

$$(1-z^2)p'(z) + (Nz - \tfrac{1}{2}\beta_1)p(z)$$
$$= (P_{N-1} - \tfrac{1}{2}\beta_1 P_N) + (-(N-1)P_1 + NP_1 - \tfrac{1}{2}\beta_1 P_0)z^N$$
$$+(-NP_0 + NP_0)z^{N+1}$$
$$= 0$$

using (3.8*a*), (3.9*a*). So

$$p(z) = (1-z)^m(1+z)^{N-m} \qquad m = \tfrac{1}{2}(N - \tfrac{1}{2}\beta_1). \tag{3.10}$$

Conditions (3.8*a*) are easily checked. Requiring $p(z)$ to be a polynomial, or equivalently that (3.9*a*) be satisfied, yields

$$m \in \{0, 1, 2, \ldots, N\} \tag{3.11}$$

so that

$$\beta_1 = \begin{cases} \pm 2, \pm 6, \ldots, \pm 2N & N \text{ odd} \\ 0, \pm 4, \pm 8, \ldots, \pm 2N & N \text{ even.} \end{cases} \tag{3.12}$$

In particular, we have that β_1 is a constant, so that $\mathrm{d}P_k/\mathrm{d}x = 0$ for all k.

Consider next the relations for S_k, and set

$$s(z) = S_{N-1} + S_{N-2}z + \cdots + S_0 z^{N-1}.$$

The condition (3.9*b*) is equivalent to $S_{-1} = 0$, which together with $P_k = 0$ for $k < 0$ gives $S_k = 0$ for $k < 0$, while (3.8*a*), (3.8*b*) gives $S_k = 0$ for $k > N$. Then by an argument similar to that for $p(z)$ and by using (3.8*b*), (3.9*b*), we have

$$2z(1-z^2)s'(z) + [2(N-1)z^2 - \beta_1 z + 1]s(z)$$
$$= \beta_0 p(z) + (S_{N-1} - \beta_0 P_N) + (S_1 - \beta_1 S_0 - \beta_0 P_0)z^N$$
$$+\big(-2(N-1)S_0 + 2(N-1)S_0\big)z^{N+1}$$
$$= \beta_0 p(z).$$

Therefore

$$s(z) = \tfrac{1}{2}\beta_0 p(z) z^{-1/2}(1-z^2)^{-1/4}\int_0^z z^{-1/2}(1-z^2)^{-3/4}\,\mathrm{d}z$$

$$= \beta_0 p(z) F(1, \tfrac{1}{2}; \tfrac{5}{4}; z^2)$$

where F is the usual hypergeometric function, and we have used the following well known formulae (cf [19])

$$\int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t = p^{-1}x^p F(p, 1-q; p+1; x)$$

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

Note that $s(0) = S_{N-1} = \beta_0$ so (3.8b) is satisfied. Now the only way that $s(z)$ can be a polynomial of degree $N-1$ is if

$$\beta_0 \equiv 0$$

yielding $S_k = 0$ for $k = 0, \ldots, N$. Indeed, in order for $s(z)$ to be such a polynomial, $F(1, \frac{1}{2}; \frac{5}{4}; \zeta)$ must be of the form $(1-\zeta)^\mu p_n(\zeta)$ where p_n is a polynomial of degree n, $p_n(1) \neq 0$ and $2\mu + 2n \leqslant -1$ (cf [19, section 2.2.1], noting that $F(a, b; c; 0) = 1$). Using the identity, ([19, p 112, equation (20)], with $a = 1$, $b = \frac{1}{2}$ and $\frac{1}{2}(a+b+1) = \frac{5}{4}$) leads to a contradiction.

Next, we assume the above results for β_1, P_k, β_0 and S_k, and consider the relations for T_k. Set

$$t(z) = T_{N-1} + T_{N-2}z + \cdots + T_0 z^{N-1}$$

and note that $T_k = 0$ for $k < 0$ and $k \geqslant N$. Then a similar calculation to that for $p(z)$ and $s(z)$ above yields

$$(1-z^2)t'(z) + ((N-1)z - \tfrac{1}{2}\beta_1)t(z) = \tfrac{1}{2}x\left(zp'(z) - Np(z)\right).$$

Thus

$$t(z) = p(z)(1-z^2)^{-1/2}\left[H_{N-1}(x) - x\beta_1/4\right] + \tfrac{1}{2}xp(z)(1-z^2)^{-1}(\tfrac{1}{2}\beta_1 - Nz).$$

It is easy to check that the second summand is a polynomial of degree $N-1$ if (3.10) and (3.11) hold. Therefore, for $t(z)$ to be a polynomial, we must have that

$$H_{N-1}(x) = \tfrac{1}{4}\beta_1 x.$$

Finally, setting

$$f(z) = F_{N-2}z + F_{N-3}z^2 + \cdots + F_0 z^{N-1}$$

using methods similar to those above and using all the results obtained above, we have

$$2z(1-z^2)f'(z) + (2(N-1)z^2 - \beta_1 z - 1)f(z) = \alpha z^2 p'(z) - \alpha N z p(z) - z t_x(z)$$

which has the solution (choosing the constant of integration to be zero so that $f(z)$ is analytic at the origin)

$$f(z) = -\left(\tfrac{1}{4}\beta_1 + \alpha N\right) zp(z)(1-z^2)^{-1} + \tfrac{1}{6}\left(\alpha\beta_1 + N\right) z^2 p(z)(1-z^2)^{-1} F(1, \tfrac{1}{2}; \tfrac{7}{4}; z^2).$$

Requiring $f(z)$ to be a polynomial of degree $N-1$ yields

$$\alpha\beta_1 + N = 0 \tag{3.13}$$

since $F(1, \frac{1}{2}; \frac{7}{4}; \zeta)$ is absolutely convergent at $|\zeta| = 1$. Equation (3.13) is the formula connecting α and N that was sought. Note that if $\beta_1 = 0$, equation (3.13) reduces to an inconsistency. Further, given (3.10) and (3.13), we have in the cases $m = 0, N$ that $\beta_1 = \pm 2N$, $\alpha = \mp\frac{1}{2}$ and $f(z) \equiv 0$.

Examination of (3.12) and (3.13) shows that

$$|\alpha| = \frac{N}{|\beta_1|} \geqslant \frac{1}{2}.$$

By use of the methods in section 3.2, if we obtain an integral for Φ_α, we must obtain one for $\Phi_{\pm\alpha\pm n}$ for all $n \in \mathbb{Z}$, so that we also have $|\pm\alpha \pm n| \geqslant \frac{1}{2}$, for all $n \in \mathbb{Z}$, and hence

$$\alpha = \tfrac{1}{2}(2n+1) \qquad n \in \mathbb{Z}$$

are the only possibilities. Since we know from the arguments in section 3.2 that special integrals do exist for these values of α, the theorem is proven. □

Note that the set of possibilities given by $N = 4n$, $n \in \mathbb{N}$ and $\beta_1 = \pm 4$, which might lead to integrals for Φ_n, has one integer missing, namely $n = 0$, which would require $N = 0$. Indeed, for example, the equations $\mathcal{C}$ in the case $N = 8$, $\beta_1 = 4$ are inconsistent. This ties in with the fact that the known rational solutions for α an integer are actually 'singular' solutions; the known solution $y = 1/x$ for $\alpha = -1$ is actually a solution of both $y'' - 2y^3 = 0$ and $xy - 1 = 0$.

It follows from the theorem that any Bäcklund transformation for the second Painlevé equation, whose action on the parameter α is such as to not preserve the set, $\{(2n+1)/2 \mid n \in \mathbb{Z}\}$, cannot be a rational expression with respect to y and y'.

4. Calculations for the fourth Painlevé equation

This example shows that integrals other than of polynomial type can arise naturally out of the calculation. The fourth Painlevé equation is given by

$$yy'' = \tfrac{1}{2}(y')^2 + \tfrac{3}{2}y^4 + 4xy^3 + 2(x^2 - \alpha)y^2 + \beta. \tag{4.1}$$

In the notation of section 2, we have $A = y$, $R = 2$ and $w_R = \frac{1}{2}$, so reading off the leading-order constraints from table 1 we have for $M = 1$ that $yq_{N,y} + Nq_N/2 = b_1 q_N$. Requiring a polynomial solution to this equation necessitates b_1 to be a constant, with

$$q_N = f(x)y^\ell \qquad \ell + \tfrac{1}{2}N = b_1$$

and where $f(x)$ is an arbitrary function and $\ell \in \mathbb{N}$. The other case, $M = 0$, leads to the equation $yq_{N,y} + Nq_N/2 = 0$, with solution $q_N = k(x)y^{-N/2}$ which is not of polynomial type. So set

$$b(x, y, y') = b_0(x, y) + b_1 y'.$$

Then inserting (2.1) into (2.7) and setting coefficients of powers of y' to zero yields the following equations:

$$yq_{N-1,y} + \left(\tfrac{1}{2}N - \tfrac{1}{2} - b_1\right) q_{N-1} = b_0 q_N - yq_{N,x} \qquad (4.2_{N-1})$$

$$\vdots$$

$$yq_{k-1,y} + \left(\tfrac{1}{2}(k-1) - b_1\right) q_{k-1}$$
$$= b_0 q_k - yq_{k,x} - (k+1)(\tfrac{3}{2}y^4 + 4xy^3 + 2(x^2-\alpha)y^2 + \beta)q_{k+1} \qquad (4.2_{k-1})$$

$$\vdots$$

$$0 = yq_{0,x} - b_0 q_0 + (\tfrac{3}{2}y^4 + 4xy^3 + 2(x^2-\alpha)y^2 + \beta)q_1. \qquad (4.2_{cc})$$

Denoting the right-hand side of equation (4.2_{k-1}) by $\sum \gamma_j(x)y^j$, then one has after integrating this equation that

$$q_{k-1} = \sum \frac{\gamma_j(x)y^j}{j-p} + y^p H_{k-1}(x) + \gamma_p(x)\log(y)$$

where $p = (N+1-k)/2$. Thus, if $p \notin \mathbb{N}$, we must have $H_{k-1}(x) = 0$, while if $p \in \mathbb{N}$, $\gamma_p(x)$ must be zero in order to stay within the polynomial domain. Thus the calculation differs from that for the second Painlevé equation, in that we obtain conditions additional to those obtained from (4.2_{cc}).

In general, there seems no real reason to exclude fractional powers and logarithmic terms. However for the fourth Painlevé equation, keeping such terms still leads only to integrals of polynomial type, since the coefficients of the additional terms become zero when the consistency conditions obtained from (4.2_{cc}) are satisfied. Presumably, this is a consequence of the equation possessing the Painlevé property, which rules out algebraic and logarithmic singularities and branch points in the solutions. For more general equations however, this will not be the case.

Examining the possibilities for the degree m of b_0 with respect to y yields the cases

(i) $m = 2$ and N arbitrary
(ii) $m = 1$ and N even
(iii) $m = 0$ and $N > 1$.

Lemma 4.1. It is sufficient to assume $q_N = 1$ with $b_1 = \frac{1}{2}N$.

Proof. The same calculation as in lemma 3.1 shows that $(g(x)y^s)Q$ satisfies the requisite equation with the coefficient b being transformed to $g'(x)y/g(x) + sy' + b$. Since the new b is of polynomial type with respect to y and y', we have that the dependence of q_N on x and y can be absorbed into b, provided that y^sQ is of polynomial type. The equations (4.2) for the q_k imply that any multiplication of q_N by y^ℓ leads only to each q_k being multiplied by y^ℓ, so it will be the case that considering $q_N = 1$ is sufficient. □

4.1. The simplest cases

The case N=1. One obtains

$$Q = y' \pm (y^2 + 2xy - 2\alpha) + 2$$

together with

$$b(x, y, y') = \tfrac{1}{2}y' \pm (\tfrac{3}{2}y^2 + xy + \alpha) - 1$$

satisfying $yQ' - bQ = yy'' - \big((y')^2/2 + 3y^4/2 + 4xy^3 + 2(x^2 - \alpha)y^2 + \beta\big)$, under the constraint

$$\beta = -2(\pm\alpha - 1)^2.$$

This leads to solutions of P_{IV} in terms of Weber–Hermite functions, [20].

The case N=2. In this case, there are three possibilities for b, namely

$$b = y' + \sum_{k=0}^{m} \beta_k(x)y^k \qquad m \in \{0, 1, 2\}.$$

In addition to the square of the two integrals found in the $N = 1$ case above we obtain for $m = 0$ the integral

$$Q = (y')^2 + 4y' - y^4 - 4xy^3 - 4(x^2 - \alpha)y^2 + 4$$

with $b = y' - 2$, valid for $\beta = -2$ and arbitrary α.

The case N=3. In addition to the cubic of the integrals obtained in the $N = 1$ case above, we obtain integrals for the parameter values $\beta = -2(\pm\alpha - 3)^2$, for which

$$\begin{aligned} Q = (y')^3 &+ \big(\pm y^2 \pm 2xy \mp 2\alpha + 6\big)(y')^2 \\ &- \big(4\alpha^2 \mp 24\alpha + 36 + y^4 + 4xy^3 - 4y^2\alpha \mp 4y^2 + 4y^2x^2 - 8yx\alpha \pm 8xy\big)(y') \\ &- 8x\big(3\alpha^2 - 14\alpha \pm 15\big)y + 4\big(\pm 6\alpha x^2 \mp 3\alpha^2 - 10x^2 + 10\alpha - 11\big)y^2 \\ &- 8x\big(\mp 3\alpha + 3 \pm x^2\big)y^3 \pm 8\alpha^3 - 72\alpha^2 \pm 216\alpha - 216 \\ &- 2\big(1 \mp 3\alpha \pm 6x^2\big)y^4 \mp 6xy^5 \mp y^6 \end{aligned}$$

and $b = 3y'/2 \pm (3y^2/2 + xy + \alpha) - 3$.

4.2. General results

An extensive discussion of Bäcklund transformations for the fourth Painlevé equation can be found in [21] and references therein. Using these Bäcklund transformations, with the method of section 3.2, one obtains integrals of the fourth Painlevé equation, for the parameter sets

$$\beta = -2n^2 \qquad n \text{ an integer and } \alpha \text{ arbitrary}$$

$$\beta = -2(\pm\alpha - m)^2 \qquad m \text{ an odd integer and } \alpha \text{ arbitrary}$$

whose degrees with respect to y' are $2n$ and m, respectively.

For the parameter values, $\beta = -2m^2$, $\alpha = 0$ and m an odd integer, there are two distinct integrals of polynomial type. These integrals are related by the Bäcklund transformation

$$y(x; -\alpha, \beta) = -\mathrm{i}y(\mathrm{i}x; \alpha, \beta).$$

5. The sixth Painlevé equation

This example shows how the method may be modified if the integration of the associated system for the q_k is difficult to do in closed form. The sixth Painlevé equation is

$$A(x, y)y'' = W_2(x, y)(y')^2 + W_1(x, y)y' + W_0(x, y)$$

where

$$A(x, y) = y(y-1)(y-x)$$

$$W_2(x, y) = \tfrac{1}{2}(3y^2 - 2(x+1)y + x)$$

$$W_1(x, y) = -\frac{y(y-1)}{x(x-1)}\left(y(2x-1) - x^2\right)$$

$$W_0(x, y) = \frac{y^2(y-1)^2(y-x)^2}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)^2}{(y-x)^2}\right).$$

Similar arguments as for the second and fourth Painlevé equations lead to the facts that one may set $q_N \equiv 1$ together with $b = NW_2y' + b_0(x, y)$, where degree$(b_0, y) = 4$, unless N is even in which case one may take degree$(b_0, y) \leqslant 4$. The associated system of equations to solve for the coefficients q_k of $(y')^k$ in Q is

$$y(y-1)(y-x)q_{N-1,y} - W_2q_{N-1} = b_0q_N - NW_1q_N \qquad (5.1_{N-1})$$

$$\vdots$$

$$y(y-1)(y-x)q_{k,y} + (k-N)W_2q_k$$
$$= b_0q_{k+1} - y(y-1)(y-x)q_{k+1,x} - (k+1)W_1q_{k+1} - (k+2)W_0q_{k+2} \qquad (5.1_k)$$

$$\vdots$$

$$0 = b_0q_0 - y(y-1)(y-x)q_{0,x} - W_0q_1\,. \qquad (5.1_{cc})$$

First, from (5.1_{N-1}) we have both the degree(q_{N-1}, y) = degree$(b_0, y) - 2$, and also the equations for the coefficients of powers of y in the symbolic expression for q_{N-1}. From successive equations in (5.1), at every stage the degree(q_k, y) can be ascertained, along with the equations for the coefficients of powers of y in the expression for q_k, in terms of the coefficients of the powers of y in the $q_j,\ j > k$.

This leads to a set of equations for the coefficients, whose solution can be obtained by the use of computer algebra to systematically select factors, simplify the nonlinear overdetermined systems of differential equations, for example using the differential algebra package `diffgrob2` [26], and perform the remaining integrations. Carrying out the calculations leads to simple algebraic or differential equations to be solved, but the number of subcases to be considered becomes large. Indeed, for $N = 2$, there are over 20 'branches' of the calculation (caused by choosing factors) to be followed! Thus, we present here the simplest result, and will pursue the application of computer algebra to this problem elsewhere.

We set $\alpha = 2a^2$, $\beta = -2b^2$, $\gamma = 2c^2$ for convenience. Then for $N = 1$, the integral obtained is

$$Q = y' - \frac{2b}{x-1} - 2\left(a + x(c-b) - c\right)\frac{y}{x(x-1)} + 2a\frac{y^2}{x(x-1)}$$

subject to the parameter condition

$$\delta = -2(a+b+c+1)(a+b+c)$$

for all choices of the signs of a, b and c. Solutions of the sixth Painlevé equation can be obtained from these special integrals in terms of the hypergeometric function [22–24].

6. Integral of a third-order equation

In this section, we extend the method to find the *general* first integral of the third-order ordinary differential equation,

$$x^2FF''' = (x^2F' - xF)F'' - F^3F' - 2\mu_1(xF - x^2F') + \mu_2 xF^2 \qquad (1.1)$$

which arises as a classical reduction of a (2+1)-dimensional sine-Gordon system [1]. The equation (1.1) has no classical or contact symmetries, with which the integral could be calculated. We show here that the general first integral, which is a rational expression in x, F, F' and F'', can be obtained by the method discussed in this paper. Indeed, the constant of integration appears as a constant of integration of one of the subsidiary equations.

We thus seek to solve the equation

$$\begin{aligned} x^2FQ_x + x^2FF'Q_F + x^2FF''Q_{F'} + [(x^2F' - xF)F'' + \widetilde{W}(x, F, F')]Q_{F''} \\ = B(x, F, F', F'')Q \end{aligned} \qquad (6.1)$$

where

$$\widetilde{W} = -F^3F' - 2\mu_1(xF - x^2F') + \mu_2 xF^2 \qquad Q = \sum_{k=0}^{N} q_k(x, F, F')(F'')^k$$

and where the q_k are polynomial with respect to both F and F'. Computing possible powers of B with respect to F'' yields $\deg(B, F'') = 0$, so that setting coefficients of powers of F'' in (6.1) to zero yields the system of equations

$$q_{N,F'} = 0 \qquad (6.2_N)$$

$$x^2F\,q_{N-1,F'} = Bq_N - x^2Fq_{N,x} - x^2FF'q_{N,F} - N(x^2F' - xF)q_N \qquad (6.2_{N-1})$$

$$\vdots$$

$$\begin{aligned} x^2Fq_{N-k-1,F'} = Bq_{N-k} - x^2Fq_{N-k,x} - x^2FF'q_{N-k,F} \\ -(N-k)(x^2F' - xF)q_{N-k} - (N-k+1)\widetilde{W}q_{N-k+1} \end{aligned} \qquad (6.2_{N-k-1})$$

$$\vdots$$

$$0 = x^2Fq_{0,x} + x^2FF'q_{0,F} + \widetilde{W}q_1 - Bq_0. \qquad (6.2_{cc})$$

As before, we set $B = \sum b_j(F')^j$, and integrate each equation $(6.2_N), \ldots, (6.2_0)$ with respect to F' starting from the top down. This amounts to integrating a polynomial in F' with symbolic coefficients. Then the final equation (6.2_{cc}) yields, after setting coefficients of powers of F' to zero, a set of consistency conditions for the b_j and the functions of integration. It is not difficult to show, by comparing powers of F' in the final equation, that $\deg(B, F') = 1$. So, set

$$B = b_0(x, F) + b_1(x, F)F'$$

and perform the integration just outlined, with $N = 2$. (This is the value of N for which we obtain a solution.) The first consistency condition, the coefficient of $(F')^5$ in (6.2_{cc}), is

$$-x^2Fq_2b_{1,F} + 3x^6F^2q_{2,FF} + 4x^4q_2b_1 - 3x^6Fq_F - 3x^4F^2q_{2,FF}b_1 + x^6F^3q_{2,FFF}$$
$$+3x^2Fq_2b_1b_{1,F} + 3x^2Fb_1^2q_{2,F} - 3x^4F^2q_{2,F}b_{1,F} - x^4F^2q_2b_{1,FF}$$
$$-b_1^3q_2 - 3x^4Fb_1q_{2,F} = 0.$$

Since we are assuming that both q_2 and b_1 are polynomial with respect to F, we have by comparing powers of F in the terms in this equation that $\deg(b_1, F) = 0$. Substituting $b_1 = s(x)$, we obtain an equation linear in q_2, in which only derivatives with respect to F appear, and which is, moreover, homogeneous of Euler type; a good sign, since we are seeking a polynomial solution. Indeed, substituting in $b_1 = s(x)$, $q_2 = t(x)F^n$, yields

$$t(x)F^n(-s(x) + nx^2)(-s(x) + (n+2)x^2)(-s(x) + (n-2)x^2) = 0.$$

The same argument as in lemma 3.1 yields that we can set $t(x) = 1$, and we carry the F^n in order to keep everything polynomial; we decide the value of n at the end.

The next condition, the coefficient of $(F')^4$ in (6.2_{cc}), yields an equation for $b_0(x, F)$ in which only derivatives with respect to F appear, and which is inhomogeneous of Euler type. Setting $b_0 = t(x)F + m(x)F^m$ yields, for $s(x) = nx^2$ that $t(x) = -3x/2$ and $m = 2, -1$; for $s(x) = (n-2)x^2$ that $t(x) = -x$ and $m = -1, -3$; and for $s(x) = (n+2)x^2$, that $t(x) = -2x$ and $m = 2, 4$. It transpires that the cases $s(x) = nx^2$, $(n-2)x^2$, lead later in the calculation to inconsistencies, so we do not consider these cases any further here.

Thus, we have that $q_2 = F^n$, $b_1 = (n+2)x^2$, and $b_0 = -2xF + m_1(x)F^2 + m_2(x)F^4$. The next consistency condition involves the function of integration $H_1(x, F)$ appearing in the expression for q_1. Again, the only derivatives of H_1 occurring are with respect to F, and regarded as a condition for H_1, the equation is inhomogeneous of Euler type, and so is easily integrated to obtain a solution polynomial with respect to F, with coefficients some (to be determined) unknown functions of x. The next consistency condition yields conditions on the various functions of x that appear in b_0 and H_1.

Continuing in this way, we arrive at an expression for an integral of (1.1), which contains one arbitrary constant, κ, and from which F^n divides out, so we may set $n = 0$ to obtain

$$Q = (F'')^2 - 2\mu_2FF'' + 4\mu_1F'' + \frac{F^2(F')^3}{x^2} + \frac{F^2\kappa}{x^2} + \mu_2^2F^2 - \frac{\mu_2F^4}{x^2}$$
$$+4\frac{\mu_1F^3}{x^2} - 4\mu_1\mu_2 + 4\mu_1^2$$

with $B = -2xF + 2x^2F'$.

The appearance of the arbitrary constant κ, which arises as a constant of integration of one of the subsidiary conditions, leads one to suspect that we have found the general first integral, which is indeed the case.

Using the method of Bureau [25], the integral of (1.1) can be transformed to a subcase of the fifth Painlevé equation. Hierarchies of exact solutions with Bäcklund transformations connecting them for (1.1) can be found in [1].

Thus, application of the method in this case yields the general first integral of a third-order nonlinear ordinary differential equation by means of a series of linear ordinary differential equations of Euler type, which represents a significant reduction in the difficulty of the integration problem. Comparing this method with the one used in [6], one can see that both methods have to guess the degree of the integral with respect to F'', and while the method used in [6] involves *ad hoc* simplifying assumptions, we are assuming a rational ansatz.

7. Discussion

For ordinary differential equations that have no symmetries, classical or contact, to aid in their integration, we feel the method described in this paper is an interesting addition to the equation solver's 'tool box'; firstly, because of the possibility of obtaining special integrals for certain parameter values, secondly because it is algorithmic, thirdly because most of the calculation can be performed by a computer algebra package, and finally, the consistency conditions are often simple to solve. Indeed, most of the method involves integration of polynomial expressions with symbolic coefficients, selection of coefficients, and comparison of degrees of monomials, all of which are easily performed by computer algebra packages. Further, packages which can simplify over-determined systems of differential equations (cf [26]) and perform integration heuristics can be used to semi-automate the entire process.

For some examples, such as the sixth Painlevé equation (cf section 6), the integration of the relevant rational expression in the associated equations may not be expressible easily in simple form. We have shown how the method may be adapted to this case. Further, unlike the process of substituting in an arbitrary differential polynomial, our method solves for the bounds on the degrees of the various derivative terms as one proceeds, given only the chosen degree of the highest derivative term in the special integral.

As mentioned in the introduction, special integrals of the Painlevé equations can often be obtained by the isomonodromy deformation method. That method can only be applied to equations for which a Lax pair is known. The strength of the method demonstrated here is that it can be applied to a great many equations about which one knows almost nothing, since in principle it can be applied to any ordinary differential equation of polynomial type of any order but of degree one. While in this paper we have applied the method only to equations possessing the Painlevé property, we have not used the Painlevé property in any way. It is only that such equations seem to possess special integrals for certain values of their parameters that we use them here to demonstrate the method. The weakness of the method, as demonstrated here, is that a polynomial ansatz for the integral is a strong one, and extensions of the method to other function spaces are important. We have shown that some extensions arise naturally during the course of the calculation, but a method to determine the most general form of the ansatz for a particular equation is required, and would be of great interest.

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